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Order Statistics for First Passage Times in One-Dimensional Diffusion Processes

S. Bravo Yuste[1] **and Katja Lindenberg**[2]

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The problem of the statistical description of the first passage time $t_{j,N}$ to one or two absorbing boundaries of the first j of a set of N independent diffusing particles in one dimension is revisited. An asymptotic expression for large N of the generating function of the moments of $t_{j,N}$ is obtained, and explicit expressions for the first two moments are presented. The results are valid for a specific but broad class of initial distributions of particles and boundaries. The mean first passage time of the first particle $\langle t_{1,N} \rangle$ and its variance are compared with numerical estimates for an initial distribution in which all particles are placed at the midpoint of the diffusion region.

KEY WORDS: Diffusion; order statistics; trapping; mean first passage times.

1. INTRODUCTION

The problem addressed in this paper is related to the extensively studied trapping problem (see refs. 1–3 and references therein) in which a diffusing particle is absorbed by a trap. We study the order statistics of absorption times of *a set* of independent diffusing particles in one dimension for a given configuration of boundaries, that is, we give a statistical description of the time $t_{j,N}$ spent by the jth of N diffusing walkers inside an interval before being absorbed by the boundaries (traps). Usually only first-passage-time problems of a single particle are studied, whereas in reality a finite number of particles may be present simultaneously. This distinction

[1] Departamento de Fisica, Universidad de Extremadura, 06071 Badajoz, Spain.

[2] Department of Chemistry and Biochemistry and Institute for Nonlinear Science, University of California-San Diego, La Jolla, California 92093-0340.

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may be important if the first or first few particles that arrive at an absorber lead to a trigger effect.

The order statistics of a set of random walkers on infinite lattices of D dimensions (D integer) was studied by Lindenberg *et al.*[4] The order statistics of diffusing particles in one dimension, with some discussion of higher dimensions, was studied by Weiss *et al.*[5] However, some of the results given there were incorrect due to the fact that only the main term of the asymptotic expansion for large N of the generating function of the moments was calculated correctly. Here we shall rectify the error and also carry out the asymptotic expansions to higher orders in order to find the *main* term of the asymptotic expansion of the *variance* of the first passage time.

In Section 2 we give a short review of basic formulas for order statistics of diffusing particles in one dimension and we present the class of initial distributions of particles for which our results are valid. The asymptotic analysis for large N of the generating function of the moments of the first passage time is carried out in Section 3. In Section 4 we obtain explicit expressions for the first two moments of $t_{j,N}$, and a comparison with numerical estimates is carried out for particles initially placed at the midpoint of the diffusion region. The results are summarized and discussed in Section 5.

2. BASIC FORMULAS

Let $G(t)$ be the probability that a single diffusing particle has not been absorbed by the boundaries at either end of a spatial interval in the time interval $(0, t)$, and let $g(t) = -dG(t)/dt$ be the first-passage-time density. The probability density $q_{j,N}(t)$ for the absorption time of the jth out of N indistinguishable and noninteracting particles is[5,6]

$$q_{j,N}(t) = j\binom{N}{j} g(t)[1-G(t)]^{j-1} G^{N-j}(t), \qquad j = 1, 2, ..., N \tag{2.1}$$

With this expression, the generating function of $q_{j,N}$ can be written as

$$Q_N(z; t) = \sum_{j=1}^{N} q_{j,N}(t)\, z^{j-1} = Ng(t)\{G(t) + [1-G(t)]\, z\}^{N-1} \tag{2.2}$$

The mth moment of the absorption time of the jth out of N diffusing particles is

$$\mu_{j,N}(m) = \int_0^\infty t^m q_{j,N}(t)\, dt \tag{2.3}$$

and its generating function can be written by means of Eqs. (2.2) and (2.3) as

$$U_{N,m}(z)=\sum_{j=1}^{N}\mu_{j,N}(m)\,z^{j-1}=\int_0^\infty t^m Q_N(z,t)\,dt \tag{2.4}$$

Upon integration by parts, Eq. (2.4) becomes

$$U_{N,m}(z)=\frac{m}{1-z}\int_0^\infty t^{m-1}\{(G(t)+[1-G(t)]\,z)^N-z^N\}\,dt \tag{2.5}$$

We seek explicit expressions for $\mu_{j,N}(m)$ when $j<N$ and $N\gg 1$. This can be accomplished through the evaluation of this integral for large N:

$$U_{N,m}(z)\approx\frac{m}{1-z}\int_0^\infty t^{m-1}\exp\{N\ln[1-h(t)(1-z)]\}\,dt \tag{2.6}$$

The mortality function $h(t)=1-G(t)$ is the probability that the particle *has* been absorbed in the time interval $(0,t)$. In terms of the probability $h(x,t)$ that a particle that starts at x will be absorbed by the boundaries placed at $x=0$ and $x=L$ in the time interval $(0,t)$, and the initial density probability function $p(x)$,

$$h(t)=\int_0^L p(x)\,h(x,t)\,dx \tag{2.7}$$

where (cf. ref. 6, Section X.5),

$$h(x,t)=2\left[1-\phi\left(\frac{x}{\sqrt{2Dt}}\right)\right]+2\sum_{m=1}^{\infty}(-1)^m\left[\phi\left(\frac{mL-x}{\sqrt{2Dt}}\right)-\phi\left(\frac{mL+x}{\sqrt{2Dt}}\right)\right] \tag{2.8}$$

and $\phi(x)$ is the standard normal distribution.

In the remainder of this paper we express all times t in units of $L^2/8D$ and all distances in units of L, that is, times and distances are dimensionless. All other functions are suitably rescaled to dimensionless form as well. The results of the present paper are valid for all those initial distributions (and boundary conditions) that lead to a mortality function of the form

$$h(t)\approx at^{\alpha}e^{-t_0/t}(1+h_1t) \tag{2.9}$$

for t small, where a, α, t_0, and h_1 are dimensionless constants. Clearly, the associated initial distributions vanish at the boundaries since the mortality function is zero at $t=0$. We keep the term $h_1 t$ because it contributes significantly to the moments of the absorption time. The initial distribution of particles $p(x)=\delta(x-1/2)$ with absorbing boundaries at $x=0$ and $x=1$ leads to the mortality function (2.9) with $a=4/\sqrt{2\pi}$, $t_0=1/2$, $h_1=-1$, and $\alpha=1/2$. This same initial distribution with the left absorbing boundary at $x=0$ but with the right absorbing boundary placed at $x>1$ also leads to (2.9) with the same values of t_0, h_1, and α, but now with $a=\sqrt{2/\pi}$.

More generally, one finds the mortality function (2.9) for initial distributions where $p(x)=0$ when x is outside the open interval (x_1, x_2) (with $0<x_1<x_2<1$) and where $\lim_{x\to x_0} p(x)/(x-x_0)^n=p_0\neq 0$. Here x_0 stands for x_1 or x_2, and p_0 stands for p_1 or p_2.[5] In this case,

$$a=\frac{\Gamma(n+1)}{2^{2n+2}\sqrt{2\pi}}\frac{p_0}{x_0^{n+2}}, \qquad t_0=2x_0^2, \qquad h_1=-\frac{1}{4x_0^2}, \qquad \alpha=n+\frac{3}{2} \tag{2.10}$$

with $x_0=x_1$ and $p_0=p_1$ if $x_1<1-x_2$; $x_0=1-x_2$ and $p_0=p_2$ if $x_1>1-x_2$; and $x_0=x_1$ and $p_0=p_1+p_2$ if $x_1=1-x_2$. This functional form is also valid for systems with these initial distributions but with a reflecting barrier, say at $x=0$, and a trap, say at $x=1$. In this case $x_0=x_2$ and $p_0=p_2$.

3. ASYMPTOTIC EXPANSIONS

We now use the general form (2.9) to evaluate the asymptotic expansion for large N of the generating function of the moments, $U_{N,m}(z)$. We write Eq. (2.6) as

$$U_{N,m}(z)=U_{N,m}^{(\tau)}(z)+U_{N,m}^{(\infty)}(z) \tag{3.1}$$

where

$$U_{N,m}^{(\tau)}(z)\approx\frac{m}{1-z}\int_0^\tau dt\, t^{m-1}F(t,N,z) \tag{3.2}$$

$$U_{N,m}^{(\infty)}(z)\approx\frac{m}{1-z}\int_\tau^\infty dt\, t^{m-1}F(t,N,z) \tag{3.3}$$

and

$$F(t,N,z)=[1-h(t)(1-z)]^N \tag{3.4}$$

The time τ is chosen so that (i) $\tilde{h}(t) = at^{\alpha} \exp(-t_0/t)$ is a good aproximation to $h(t)$ for $0 \leqslant t \leqslant \tau$ and (ii) $F(\tau, N, z) \ll 1$ for z small, say $|z| \ll 1$. The first condition implies that $\tilde{h}(t)$ should be small so that

$$F(t, N, z) = \tilde{F}(t, N, z)[1 + \mathcal{O}(N\tilde{h}^2(t))]$$

with $\tilde{F}(t, N, z) = \exp[-N(1-z)\tilde{h}(t)]$. The terms $\mathcal{O}[N\tilde{h}^2(t)]$ can be neglected for large N (cf. below):

$$U_{N,m}^{(\tau)}(z) \approx \frac{m}{1-z} \int_0^{\tau} dt\, t^{m-1} \tilde{F}(t, N, z) \tag{3.5}$$

We estimate τ by requiring that

$$|e^{-Nh(\tau)} - e^{-N\tilde{h}(\tau)}| \simeq |h(\tau) - \tilde{h}(\tau)| N e^{-N\tilde{h}(t)} = 1/k$$

where $k \gg 1$ is a large arbitrary constant that, for convenience, we set equal to $1/|h(\tau) - \tilde{h}(\tau)|$. Therefore, we find that τ satisfies

$$\exp[-N\tilde{h}(\tau)] = 1/N \tag{3.6}$$

Approximate solution of this equation yields

$$\tau \simeq \frac{t_0}{\ln N - \ln \ln (N)} \tag{3.7}$$

It should be noted that, for large N, $\exp[-N\tilde{h}(t)]$ changes abruptly near $t = \tau$ so that the estimates of τ obtained from (3.7) may lead to inaccurate results if they are used to evaluate $\exp[-N\tilde{h}(\tau)]$. Also note that (3.6) implies $N\tilde{h}^2(\tau) = \ln^2 N/N \ll 1$ for large N.

Because $F(t, N, z)$ is a monotonically decreasing function of t and N, condition (ii), $F(\tau, N, z) \ll 1$, allows us to neglect the contributions of $U_{N,m}^{(\infty)}(z)$ to the absorption time moments compared with those of $U_{N,m}^{(\tau)}(z)$ provided that $F(t, N, z)$ goes to zero sufficiently quickly. This is a reasonable assumption, at least for the physical systems described in Section 2. For example, it is well known[1] that for the system with two absorbing barriers at 0 and L

$$h(x, t) = 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin[(2n+1)\pi x]\, e^{-(2n+1)^2 \pi^2 t} \tag{3.8}$$

For times beyond τ only a finite number of modes is important. Therefore, from Eq. (2.7), $G(t) \equiv 1 - h(t)$ is determined by a sum of terms proportional to $\exp[-(2n+1)^2 \pi^2 t]$ ($n = 0, 1, 2,...$). The contribution of $U_{N,m}^{(\infty)}$ to the moments is consequently negligible. In particular, the contribution from the interval (τ, ∞) to $\mu_{1,N}(1)$ obtained by setting $z=0$ in Eqs. (3.3) and (3.4) is a sum of terms of order $\exp[-(2n+1)^2 \tau N]$ that, taking Eq. (3.7) into account, are negligible relative to the contributions from the time interval $(0, \tau)$. From here on we assume that $U^{(\infty)}_{N,m}$ is negligible compared to $U^{(\tau)}_{N,m}$.

In order to evaluate $U^{(\tau)}_{N,m}(z)$ we write Eq. (3.5) as

$$U^{(\tau)}_{N,m}(z) = m t_0^m \frac{f_m(\lambda)}{1-z} \tag{3.9}$$

where

$$f_m(\lambda) \approx \int_0^{\tau/t_0} dx\, x^{m-1} \exp[-\rho\lambda(1+t_0 h_1 x)] \tag{3.10}$$

$$\lambda = N a t_0^\alpha (1-z), \qquad \rho \equiv \rho(x) = x^\alpha e^{-1/x} \tag{3.11}$$

Using ρ as the variable of integration and writing the exponential term as the series

$$\exp[-\lambda\rho(1+t_0 h_1 x)] = \exp(-\lambda\rho)(1-\lambda\rho t_0 h_1 x + \cdots)$$

(which converges for $0 \leqslant x \leqslant \tau/t_0$ when N is large), one finds that Eq. (3.10) becomes

$$f_m(\lambda) \approx f_m^{(0)}(\lambda) - h_1 f_m^{(1)}(\lambda) \tag{3.12}$$

where

$$f_m^{(n)}(\lambda) = \lambda^n t_0^n \int_0^\varepsilon \frac{d\rho}{\rho^{1-n}} e^{-\lambda\rho} \frac{x^{m+n+1}(\rho)}{1+\alpha x(\rho)} \tag{3.13}$$

with $\varepsilon = (\tau/t_0)^\alpha \exp(-t_0/\tau)$. It is important to note that Eq. (3.7) implies $\tilde{h}(\tau) = \ln(N)/N \ll 1$ and $N\tilde{h}(\tau) = \ln(N) \gg 1$, or, in terms of the new variables, $\varepsilon \ll 1$ and $\lambda\varepsilon \gg 1$.

We shall evaluate these integrals by means of the procedure used in ref. 5. In order to obtain the function $x(\rho)$, we take logarithms of both sides of Eq. (3.11) and write $u = x^{-1} - \alpha \ln x$, where $u = -\ln \rho$. We invert this relation, $x(\rho) = [u + \xi(\rho)]^{-1}$, and note that the function $\xi(\rho)$ satisfies $\lim_{\rho \to 0}[\xi(\rho)/u] = 0$. Neglecting terms $\mathcal{O}(\xi^3/u^3)$, we find

$$\alpha \ln u + (\alpha + u)\frac{\xi}{u} - \frac{\alpha}{2}\left(\frac{\xi}{u}\right)^2 = 0 \tag{3.14}$$

and, neglecting the quadratic term ξ^2/u^2, we get

$$\frac{\xi}{u} = -\alpha u^{-1} \ln u + \mathcal{O}(u^{-2} \ln u) \tag{3.15}$$

It should be noted that this first-order solution is the *negative* of the one given in ref. 5. This has important consequences. For example, the correction terms in the asymptotic expansion of the first-passage-time moments $\mu_{j,N}(m)$ differ from those given in ref. 5; this implies that the *main* term of the variance is also different. In the rest of this paper we shall display the asymptotic expansions up to the order necessary to calculate the main term of the variance. The solution of the quadratic equation (3.14) that agrees with the first-order solution (3.15) is

$$\frac{\xi}{u} = -\alpha u^{-1} \ln u + \alpha^2 u^{-2} \ln u + \mathcal{O}(u^{-3} \ln^3 u) \tag{3.16}$$

With this we then find

$$\begin{aligned}\frac{x^{\gamma}(\rho)}{1+\alpha x(\rho)} = u^{-\gamma} \Bigg[& 1 + \gamma\alpha u^{-1} \ln u - \alpha u^{-1} + \gamma \left(1 + \frac{\gamma - 1}{2}\right) \alpha^2 u^{-2} \ln^2 u \\ & - (2\gamma + 1)\, \alpha^2 u^{-2} \ln u + \alpha^2 u^{-2} + \mathcal{O}(u^{-3} \ln^3 u) \Bigg]\end{aligned} \tag{3.17}$$

Inserting this result into Eq. (3.13) and neglecting the contribution from terms of order $u^{-3-\gamma} \ln^3 u$ (where $\gamma = n+2$ for $m = 1$ and $\gamma = n+3$ for $m = 2$) or smaller, we find

$$f_1^{(0)}(\lambda) = I_2 + 2\alpha J_3 - \alpha I_3 + 3\alpha^2 K_4 - 5\alpha^2 J_4 + \alpha^2 I_4 \tag{3.18}$$

$$f_2^{(0)}(\lambda) = I_3 + 3\alpha J_4 - \alpha I_4 + 6\alpha^2 K_5 - 7\alpha^2 J_5 - \alpha^2 I_5 \tag{3.19}$$

where

$$I_\mu = \int_0^\varepsilon \frac{d\rho}{\rho} e^{-\lambda\rho} \frac{1}{\ln^\mu(1/\rho)} \approx \frac{1}{\mu - 1} \frac{1}{\ln^{\mu-1} \lambda} - \frac{C}{\ln^\mu \lambda} + \frac{\mu}{2} \frac{B}{\ln^{\mu+1} \lambda} \tag{3.20}$$

$$J_\mu = \int_0^\varepsilon \frac{d\rho}{\rho} e^{-\lambda\rho} \frac{\ln[\ln(1/\rho)]}{\ln^\mu(1/\rho)} \approx \frac{1/(\mu-1) + \ln \ln \lambda}{(\mu - 1) \ln^{\mu-1} \lambda} - \frac{C \ln \ln \lambda}{\ln^\mu \lambda} \tag{3.21}$$

$$K_\mu = \int_0^\varepsilon \frac{d\rho}{\rho} e^{-\lambda\rho} \frac{\ln^2[\ln(1/\rho)]}{\ln^\mu(1/\rho)} \approx \frac{1}{\ln^{\mu-1} \lambda} \left[\frac{2}{(\mu-1)^3} + \frac{2 \ln \ln \lambda}{(\mu-1)^2} + \frac{\ln^2 \ln \lambda}{\mu - 1} \right] \tag{3.22}$$

Here $C \simeq 0.577215$ is the Euler constant and $B = \pi^2/6 + C^2$. The integrals I_μ and J_μ were evaluated in ref. 5, although the last terms in Eqs. (3.20) and (3.21) were not computed there. The method for evaluating K_μ is similar. For $n = 1$, we obtain

$$f_1^{(1)}(\lambda) = 3t_0 I_4, \qquad f_2^{(1)}(\lambda) = 4t_0 I_5 \tag{3.23}$$

With Eqs. (3.9), (3.12), and (3.18)–(3.23), one finds the generating function of the moments of first order ($m = 1$) and second order ($m = 2$):

$$U_{N,1} \approx \frac{1}{1-z}\frac{t_0}{\ln \lambda}\left\{1 + \frac{\alpha \ln \ln \lambda - C}{\ln \lambda} + \frac{1}{\ln^2 \lambda}\left[\alpha C + B - t_0 h_1 - (2\alpha C + \alpha^2) \ln \ln \lambda + \alpha^2 \ln^2 \ln \alpha\right]\right\} \tag{3.24}$$

$$U_{N,2} \approx \frac{1}{1-z}\frac{t_0^2}{\ln^2 \lambda}\left\{1 + 2\frac{\alpha \ln \ln \lambda - C}{\ln \lambda} + \frac{1}{\ln^2 \lambda}\left[2\alpha C + 3B - 2t_0 h_1 - 2(3\alpha C + \alpha^2) \ln \ln \lambda + 3\alpha^2 \ln^2 \ln \lambda\right]\right\} \tag{3.25}$$

4. FIRST-PASSAGE-TIME MOMENTS

The asymptotic values for large N of the first two moments of the first-passage-time distribution, $\mu_{j,N}(1)$ and $\mu_{j,N}(2)$, are the coefficients of the Taylor expansion in powers of z of $U_{N,1}$ and $U_{N,2}$, respectively. Therefore, the mean value of the first passage time of the *first* of N particles (N large) is (see Appendix)

$$\langle t_{1,N} \rangle \equiv \mu_{1,N}(1) \approx \frac{t_0}{\ln \lambda_0 N}\left\{1 + \frac{\alpha \ln \ln \lambda_0 N - C}{\ln \lambda_0 N} + \frac{1}{\ln^2 \lambda_0 N}\left[\alpha C + B - t_0 h_1 - (2\alpha C + \alpha^2) \ln \ln \lambda_0 N + \alpha^2 \ln^2 \ln \lambda_0 N\right]\right\} \tag{4.1}$$

where $\lambda_0 \equiv \lambda(z = 0)/N = a t_0^\alpha$. Only the main term of this expression agrees with that of ref. 5, and the corrections are nonnegligible: assuming reasonably that λ_0 is of order 1, and even with N as large as 10^{23}, the first correction term of Eq. (4.1) is about 5% of the main term. It is not difficult to prove from Eq. (3.24) that

$$\langle t_{j,N} \rangle \equiv \mu_{j,N}(1) \approx \langle t_{1,N} \rangle + \frac{t_0}{\ln^2 \lambda_0 N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{j-1}\right) \tag{4.2}$$

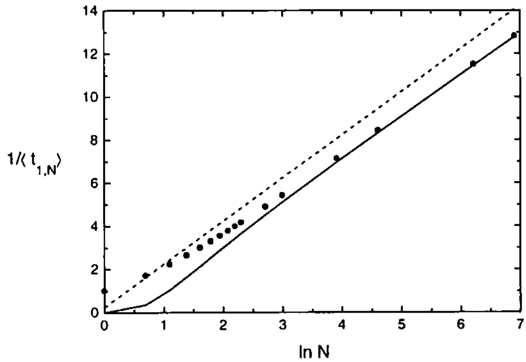

Fig. 1. Inverse of the mean first passage time $\langle t_{1,N}\rangle$ to the absorbing boundaries of the first of N particles initially placed at the midpoint of the diffusing region, plotted as a function of $\ln N$. The circles correspond to the numerical estimate and the broken and solid lines correspond to the zeroth- and second-order asymptotic formula, Eq. (4.1).

From this equation we can estimate the flux of particles absorbed by the boundaries when the jth particle is trapped:

$$\phi_j = [\mu_{j+1,N}(1) - \mu_{j,N}(1)]^{-1} \approx [\ln^2(\lambda_0 N)/t_0]\, j$$

This means that for small times the number of absorbed particles $j(t)$ grows exponentially:

$$j(t) \approx \exp\left(\frac{\ln^2 \lambda_0 N}{t_0} t\right) \tag{4.3}$$

The second moment of the first passage time of the *first* of N particles (N large) is, from Eq. (3.25) (see Appendix),

$$\begin{aligned}\langle t_{1,N}^2\rangle \equiv \mu_{1,N}(2) \approx \frac{t_0^2}{\ln^2 \lambda_0 N}\Bigg\{1 + 2\frac{\alpha \ln\ln \lambda_0 N - C}{\ln \lambda_0 N} + \frac{1}{\ln^2 \lambda_0 N}\\ \times [2\alpha(1+C) + 3B - 2t_0 h_1\\ -2(3\alpha C + \alpha^2)\ln\ln \lambda_0 N + 3\alpha^2 \ln^2 \ln \lambda_0 N]\Bigg\}\end{aligned} \tag{4.4}$$

For the jth particle, the expression is

$$\langle t_{j,N}^2\rangle \approx \langle t_{1,N}^2\rangle + \frac{t_0^2}{\ln^3 \lambda_0 N} 2\left(1 + \frac{1}{2} + \cdots + \frac{1}{j-1}\right) \tag{4.5}$$

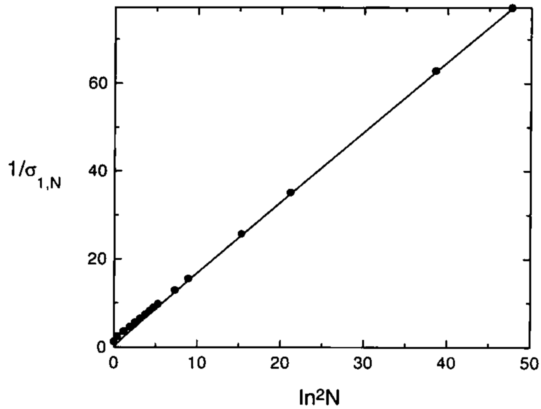

Fig. 2. Inverse of the standard deviation $\langle \sigma_{1,N} \rangle$ of the first passage time of Fig. 1, plotted as a function of $\ln^2 N$. The numerical estimate is represented by circles. The solid line corresponds to the asymptotic formula, Eq. (4.6).

From (4.1) and (4.4) one finds the variance of the first passage time of the first particle:

$$\sigma_{1,N}^2 = \langle t_{1,N}^2 \rangle - \langle t_{1,N} \rangle^2 \approx \frac{\pi^2}{6} \frac{t_0^2}{\ln^4 \lambda_0 N} \tag{4.6}$$

Figure 1 shows the mean value $\langle t_{1,N} \rangle$ of the first passage time of the first particle of a set of N particles whose initial distribution is $p(x) = \delta(x - L/2)$ obtained by numerical integration of $\langle t_{1,N}^m \rangle = m \int_0^\infty dt\, t^{m-1}[1 - h(t)]^N$ for $m = 1$. The function $h(t)$ is obtained from Eq. (2.7). we choose $L/\sqrt{2D} = 2$ so as to make $\langle t_{1,1} \rangle$ equal to unity. The value of $\langle t_{1,N} \rangle$ given by Eq. (4.1) is also shown in Fig. 1. One can see that the asymptotic formula leads to good results even for a few particles. In Fig. 2 we compare the standard deviation $\sigma_{1,N}$ with that obtained from the numerical integration of $\langle t_{1,N}^2 \rangle$. We again see that the agreement is good even for only a few particles.

5. CONCLUSIONS

We have studied the problem of the order statistics of the first passage times to certain configurations of boundaries, in one dimension, of a set of $N \gg 1$ diffusing particles when the mortality function is of the particular form (2.9) for short times. We have shown how to calculate the generating function of the first-passage-time moments of arbitrary order and obtained explicit formulas for the first two moments. This has allowed us to present explicit expressions for the mean first passage time of the jth particle $\langle t_{j,N}\rangle$ and for $\langle t_{j,N}^2\rangle$. These expressions agree with those of ref. 5 only in the leading term. As a consequence, the variance is completely different, indeed significantly smaller, than that in ref. 5.

As noted in ref. 5, the extremely mild (logarithmic in N) dependence of the mean first passage time of the first of N walkers to the ends of an interval is somewhat surprising, since one might expect the first walker to go to the boundary essentially ballistically. The reported results show that this is not the case. The nested logarithmic correction terms indicate a series that converges slowly. Similar dependences are observed in related problems, e.g., in the calculation of the span of one-dimensional multi-particle random walks.[8]

The order statistics of the diffusing process describes the rate of absorption of the diffusing particles at the boundaries. We found that initially the flow of particles grows exponentially in time with a complicated dependence on the initial number N of particles [see Eq. (4.3)]. Finally, we have shown that, for an initial distribution of particles in the form of a Dirac delta function, the asymptotic formulas obtained lead to good estimates of $\langle t_{j,N}\rangle$ and its variance even for N not too large, say $N\gtrsim 20$ for $\langle t_{j,N}\rangle$ and $N\gtrsim 10$ for the variance, when the second-order asymptotic formulas are used (see Figs. 1 and 2).

APPENDIX

In this appendix we use simple arguments to estimate the first moments $\langle t_{1,N}^m\rangle$ of the first passage time of the first particle. The key point is that the function $F(t, N, z=0)$ defined from Eq. (3.4) approximates a step function for large N: $F(t, N, 0)\approx \tilde{F}(t, N, 0)\approx \Theta(t-\tau_0)$ (Θ is the Heaviside unit step function) so that

$$\langle t_{1,N}^m\rangle \approx m\int_0^\infty dt\, t^{m-1}\tilde{F}(t, N, 0)=\tau_0^m$$

We estimate τ_0 as the time for which $\tilde{F}(\tau_0, N, 0)=1/2$. Assuming that τ_0 is small enough in order that $F(t, N, 0)\approx \exp[-N\tilde{h}(t)]$ for $t\leqslant\tau_0$, we obtain

$$\tau_0\approx\frac{t_0}{\ln N+\ln\tau_0^\alpha+\ln a-\ln 2} \tag{A1}$$

or $\tau_0 \sim t_0/\ln N$ for large N. Inserting this last equation into the right-hand side of Eq. (A1) for large N, we obtain

$$\tau_0 \sim \frac{t_0}{\ln N}\left[1+\frac{\alpha \ln \ln N + \text{const}}{\ln N}+\cdots\right] \tag{A2}$$

With Eq. (A2) we then get an estimate of $\langle t_{1,N}^m \rangle$ that agrees with the rigorous results of Eqs. (4.1) and (4.4).

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